Name Solutions

September 24, 2009

ECE 311 Exam 1 Fall 2009

Closed Text and Notes

1) Be sure you have 12 pages.

2) Write only on the question sheets. Show all your work. If you need more room for a particular problem, use the reverse side of the same page.

3) calculators allowed

4) Write neatly, if your writing is illegible then print.

5) The last 2 pages contain equations that may be of use to you.

6) This exam is worth 100 points.

(5 pts) 1. Find the distance between the two points A($1, \frac{3}{2}, \frac{1}{\sqrt{2}}$) in Cartesian coordinates and B($1, \frac{\pi}{4}, \frac{\pi}{4}$) in spherical coordinates.

Convert point B to cartesian coordinates

$\rho = r \sin\theta = 1 \sin 45° = \frac{1}{\sqrt{2}}$

$x = \rho \cos\phi = \frac{1}{\sqrt{2}} \cos 45° = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2}$

$y = \rho \sin\phi = \frac{1}{\sqrt{2}} \sin 45° = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2}$

$z = r \cos\theta = 1 \cos 45° = \frac{1}{\sqrt{2}}$

$B(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$= (1 - \frac{1}{2})^2 + (\frac{3}{2} - \frac{1}{2})^2 + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})^2$

$= \frac{1}{4} + 1 = \frac{5}{4}$

$d = \frac{\sqrt{5}}{2} = 1.118$

(5 pts) 2. The intersection of the surfaces r = 1m and $\theta = \frac{\pi}{3}$ is

a) a circle
b) a straight line
c) a sphere
d) a cone
e) c and d
f) a and b

(5 pts) 3. Find a unit vector perpendicular to the vectors $\mathbf{A} = 1\hat{a}_x + 3\hat{a}_y + 1\hat{a}_z$ and $\mathbf{B} = -2\hat{a}_x + 4\hat{a}_z$

$\vec{A} \times \vec{B} =$ a vector perpendicular to $\vec{A}$ and $\vec{B}$

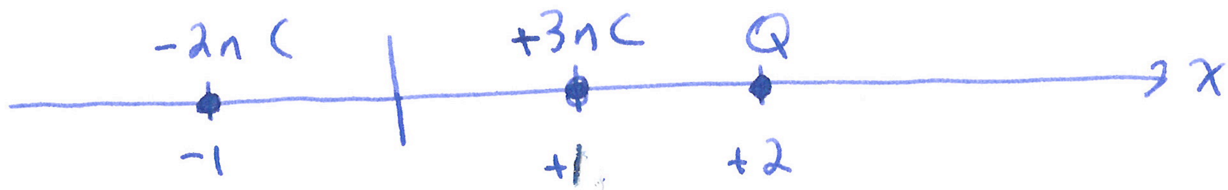
$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & 3 & 1 \\ -2 & 0 & 4 \end{vmatrix} = \hat{a}_x(12-0) - \hat{a}_y(4+2) + \hat{a}_z(0+6)$$

$$= 12\hat{a}_x - 6\hat{a}_y + 6\hat{a}_z$$

$$\text{unit vector} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{12\hat{a}_x - 6\hat{a}_y + 6\hat{a}_z}{\sqrt{(12)^2 + (6)^2 + (6)^2}} = \frac{12\hat{a}_x - 6\hat{a}_y + 6\hat{a}_z}{\sqrt{216}}$$

$$\boxed{\text{Unit vector} = 0.82\,\hat{a}_x - 0.41\,\hat{a}_y + 0.41\,\hat{a}_z}$$

(10 pts) 4. A -2 nC point charge is located at (-1, 0, 0) and a 3 nC point charge at (1, 0, 0). What charge would have to be placed at (2, 0, 0) in order for $\mathbf{E}(0, 0, 0) = 0$?



$$\vec{E}(0,0,0) = \frac{1}{4\pi\epsilon_0}\frac{-2nC}{(1m)^2}\hat{a}_x + \frac{1}{4\pi\epsilon_0}\frac{+3nC}{(1m)^2}(-\hat{a}_x) + \frac{1}{4\pi\epsilon_0}\frac{Q}{(2m)^2}(-\hat{a}_x)$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{-2nC}{1m^2} - \frac{3nC}{1m^2} - \frac{Q}{4m^2}\right]\hat{a}_x = 0$$

$$-\frac{5nC}{1m^2} - \frac{Q}{4m^2} = 0$$

$$\boxed{Q = -20nC}$$

(10 pts) 5. A cylindrical surface defined by $\rho = 1\text{m}$, $0 < z < 1\text{m}$, and has a surface charge density given by $\rho_s = z^2 \ \frac{C}{m^2}$ (when z is in meters and ϕ in radians.) What is the total charge on the surface of this cylinder?

$$Q = \int \rho_s \, ds = \int_0^1 \int_0^{2\pi} z^2 \rho \, d\phi \, dz \quad C$$

$$= \int_0^1 \int_0^{2\pi} z^2 (1) \, d\phi \, dz \quad C$$

since $\rho = 1$

$$= \int_0^1 z^2 \left[\int_0^{2\pi} d\phi \right] dz \quad C$$

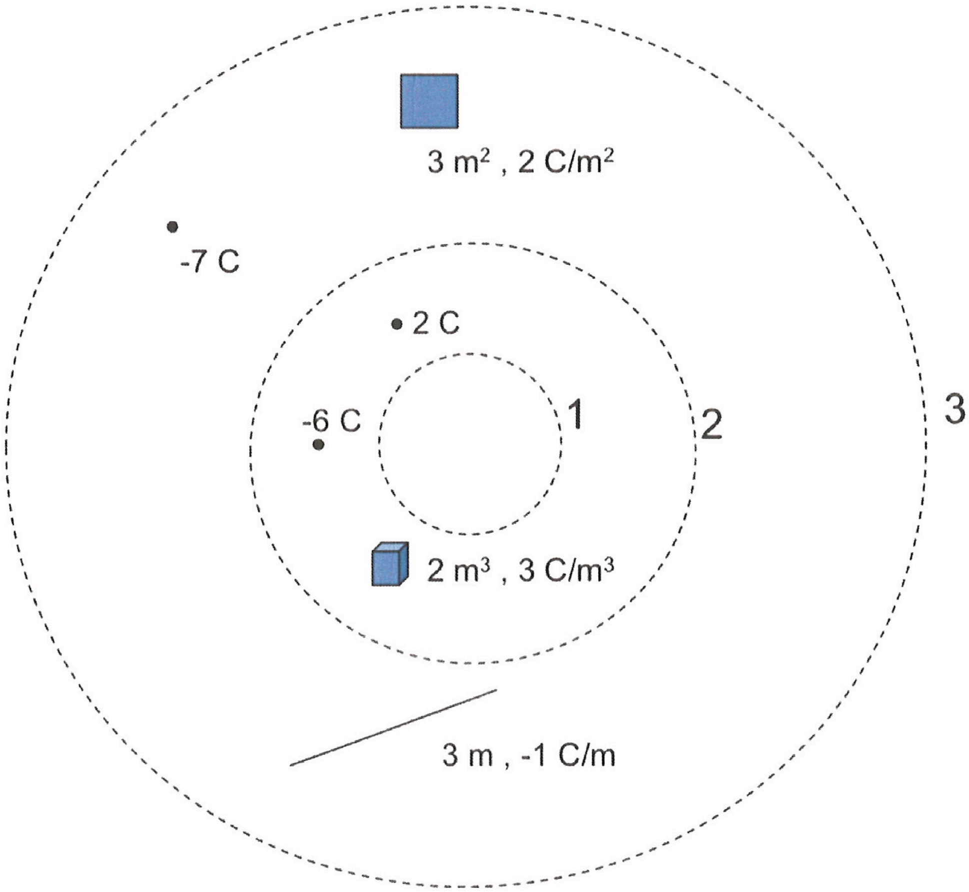
$$= \int_0^1 z^2 \left[\phi \Big|_0^{2\pi} \right] dz \quad C$$

$$= 2\pi \int_0^1 z^2 \, dz \quad C$$

$$= 2\pi \left. \frac{z^3}{3} \right|_0^1 \quad C$$

$$\boxed{Q = \frac{2\pi}{3} \quad C}$$

(12 pts) 6. In the following figure the dashed lines represent closed spherical surfaces that completely surround any objects shown within. Determine the following integrals.



$\oint \mathbf{D} \bullet d\mathbf{S}$ over surface 1 = 0

$\oint \mathbf{D} \bullet d\mathbf{S}$ over surface 2 = $2C - 6C + 2m^3\left(3\frac{C}{m^3}\right) = 2C$

$\oint \mathbf{D} \bullet d\mathbf{S}$ over surface 3 = $\oint_2 \vec{D} \cdot \vec{ds} + (3m)\left(-1\frac{C}{m}\right) - 7C + 3m^2\left(2\frac{C}{m^2}\right)$

$= 2C - 3C - 7C + 6C$

$= -2C$

(5 pts) 7. Two infinite sheets of charge of charge density 2 nC/m^2 are placed in the planes z = 6 m and z = 12 m. What would the charge density have to be on an infinite sheet of charge placed at z = 9m so that E = 0 for z > 12m and for z < 6 m?

a) 0 nC/m^2
b) 2 nC/m^2
c) -2 nC/m^2
d) 4 nC/m^2
e) -4 nC/m^2

(10 pts) 8. Given that the electric flux density is

$$\mathbf{D} = \frac{r}{3}\hat{a}_r \frac{C}{m^2} \text{ for } 0 < r \le 1m$$

note $D_\theta = 0$

$$\mathbf{D} = \frac{1}{3r^2}\hat{a}_r \frac{C}{m^2} \text{ for r > 1m}$$

$D_\phi = 0$

Find the charge density everywhere.

$$\rho_v = \nabla \cdot \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

for $0 < r \le 1m$

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{r}{3}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{3}\right) = \frac{1}{r^2} r^2 = 1 \frac{C}{m^3}$$

for $r > 1m$

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{3r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{3}\right) = 0$$

so,

$$\boxed{\rho_v = \begin{cases} 1 \frac{C}{m^3} & 0 < r \le 1m \\ 0 & r > 1m \end{cases}}$$

(10 pts) 9. Everywhere in space the electric field is $\mathbf{E} = 10\hat{a}_x \frac{V}{m}$. How much work does it take to move a 10 C charge from (10m, 10m, 10m) to (0, 0, 0)?

$$W = -Q\int_A^B \vec{E}\cdot d\vec{l} = -(10C)\int_{(10,10,10)}^{(0,0,0)} 10\hat{a}_x \frac{V}{m}\cdot(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z)$$

$$= -10C\int_{10m}^{0} 10\frac{V}{m}\,dx = (-10C)\left(10\frac{V}{m}\right)\int_{10m}^{0} dx$$

$$= (-100)\frac{C\,J}{C\,m}\int_{10m}^{0} dx = -100\frac{J}{m}\ x\Big|_{10m}^{0} = \left(-100\frac{J}{m}\right)(-10m)$$

$$\boxed{W = 1000\ J}$$

(10 pts) 10. There is a 1 nC point charge at $(r=1, \theta=\frac{\pi}{2}, \varphi=\frac{\pi}{4})$ and a 2 nC point charge at $(r=1, \theta=\frac{\pi}{3}, \varphi=\frac{\pi}{6})$. Choosing as reference V($r=\infty$) = 0, What is V(0,0,0)?

$$V(\vec{r}) = \sum \frac{Q_j}{4\pi\epsilon_0|\vec{r}-\vec{r}_j|}$$

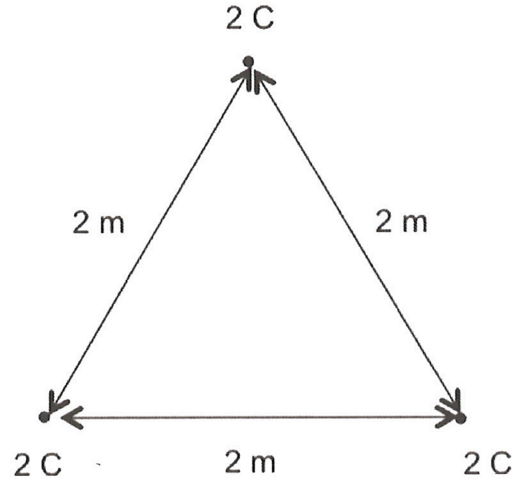
$$V(0,0,0) = \frac{1nC}{4\pi\epsilon_0(1m)} + \frac{2nC}{4\pi\epsilon_0(1m)}$$

$$= \frac{1}{4\pi\left(8.854\times10^{-12}\frac{F}{m}\right)(1m)}(3\times10^{-9}C)$$

$$= 26.96\frac{C}{F} = 26.96\frac{C}{(C/V)}$$

$$\boxed{V(0,0,0) = 26.96\ V}$$

(10 pts) 11. How much potential energy is stored in the following collection of charges? (Three 2 C point charges one at each apex of an equilateral triangle whose sides have length 2 m.)



$$W = \frac{1}{2}\sum_{k=1}^{3} Q_k V_k$$

$$V_k = \frac{1}{4\pi\epsilon_0}\frac{2C}{2m} + \frac{1}{4\pi\epsilon_0}\frac{2C}{2m}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{4C}{2m} = \frac{1}{2\pi\epsilon_0}\frac{C}{m}$$

$$W = \frac{1}{2}\left[(2C)\frac{1}{2\pi\epsilon_0}\frac{C}{m} + (2C)\frac{1}{2\pi\epsilon_0}\frac{C}{m} + (2C)\frac{1}{2\pi\epsilon_0}\frac{C}{m}\right]$$

$$= \frac{3}{2}(2C)\left(\frac{1}{2\pi\epsilon_0}\frac{C}{m}\right) = \frac{3}{2\pi(8.854\times10^{-12}\frac{F}{m})}\frac{C^2}{m}$$

$$= 5.39\times10^{10}\frac{C^2}{F} = 5.39\times10^{10}\frac{C^2}{(C/V)} = 5.39\times10^{10}\frac{C^2}{C(C/J)}$$

$$\boxed{W = 5.39\times10^{10}\ J}$$

alternate method: position the three 2C charges at $r=\infty$ and bring them in one at a time.

$$W = W_1 + W_2 + W_3$$

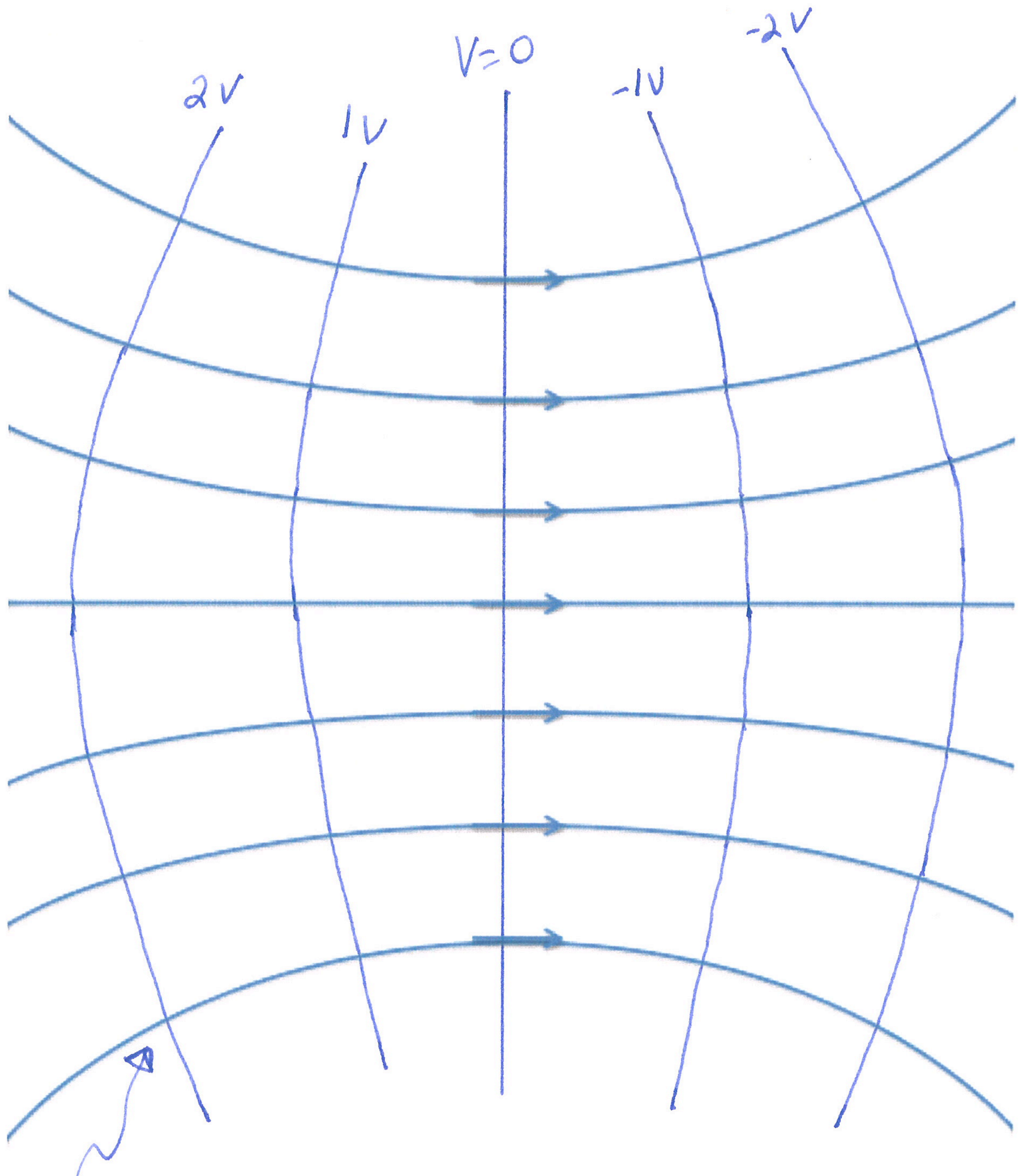
$W_1 = 0$ because it takes no work to bring in the first 2C charge

$$W_2 = QV_1 = (2C)\frac{2C}{4\pi\epsilon_0(2m)} = \frac{C^2}{4\pi\epsilon_0 m}$$

$$W_3 = QV_1 + QV_2 = 2C\frac{2C}{4\pi\epsilon_0(2m)} + 2C\frac{2C}{4\pi\epsilon_0(2m)} = \frac{4C^2}{4\pi\epsilon_0 m}$$

$$W = \frac{6C^2}{4\pi\epsilon_0 m} = 5.39\times10^{10}\ J$$

(7 pts) 12. Shown lines indicating the electric field in a region of space. Sketch five equipotential surfaces starting close to one end and finishing near the other end. Select and label one of your equitpotential surface as 0 V. Now indicate each successive surface with the appropriate voltage assuming increments of 1 V between adjacent equipotential surfaces you have drawn.



all intersections are at right angles